

Performance Optimization for an Irreversible Quantum Stirling Cooler

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The purpose of this paper is to study the optimal performance for an irreversible quantum Stirling cooler with heat leak and other irreversible losses. The relationship between the optimal cooling load and the coefficient of performance (COP) for the quantum Stirling cooler is derived. The maximum cooling load and the corresponding COP, as well as the maximum coefficient of performance and the corresponding cooling load are obtained. The experimental observation for the optimal region is provided.

Keywords: Quantum Stirling cooler, finite time thermodynamics, non-equilibrium statistical mechanics, optimization.

Introduction

The finite time thermodynamics, founded by Novikov^[1], Curzon and Ahlborn^[2], is a powerful tool for the investigation of real thermal devices. It is a discipline to study the optimum effect of thermal processes originating in finite time and the optimum performance of thermal devices and systems operating in finite time. In recent years, finite time thermodynamics has made considerable progress^[3,4]. Geva and Kosloff^[5~8] and Wu, Chen and Sun^[4,9~12] introduced a dynamical semi-group approach of quantum mechanics and non-equilibrium statistical theory into the finite time thermodynamics. It has a wide range of application for performance analysis of practical engineering cycles.

Stirling (ST) coolers have the advantages of high efficiency, low vibration levels, low noise, long service life, simple structure, etc. Therefore, they have been under investigation for the last several years for use in infrared devices and in cryogenics. Several authors have studied the finite time thermodynamics performance of the ST cooler. Wu, Chen and Sun^[11] have discussed the optimal performance for forward and reverse endoreversible quantum Stirling (QST) cycle. Based on the above work, the optimum performance for an

irreversible QST cooler with heat leak and other irreversible losses is undertaken by using the non-equilibrium statistical mechanics and the finite time thermodynamics. The fundamental relationship between the cooling load R and the coefficient of performance (COP) ε is derived in this paper. Planck's constant $h = 6.63 \times 10^{-34}$ (J·s) and $\hbar = h/2\pi = 1.05 \times 10^{-34}$ (J·s) in the International System, but $\hbar = 1$ is set in the calculation of the paper (It is equivalent to use different unit system).

Irreversible QST Cooler

The QST cooler cycle is made of two isothermal branches connected by two regenerating branches. The following assumptions are made for the QST cooler model in this paper:

(1) The working fluid of the cycle consists of many non-interacting harmonic oscillators at the atomic level. The population of the oscillator n at thermal equilibrium can be obtained from the Bose-Einstein distribution^[13]

$$n = \frac{1}{\exp(\beta\omega) - 1} \quad (1)$$

where $\omega > 0$ is the frequency of the oscillator, $\beta = 1/T$,

and T is the internal temperature of the working fluid.

(2) The cooler operates between an isothermal heat sink B_h at temperature T_h and an isothermal heat source B_c at temperature T_c . The heat source and sink are infinitely large and their internal relaxation is very strong. Thus the disturbance of the working fluid on the source or the sink can be ignored. Therefore the source B_c and the sink B_h are assumed to be in thermal equilibrium. There is heat leak Q_r between the sink B_h and the source B_c . It is the result of the coupling action between B_h and B_c by the medium of the cooler.

(3) The internal temperatures of the working fluid are T_1 and T_2 in the two isothermal processes. We assume $T_2 > T_c$ (T_c is the critical temperature) such that the Bose-Einstein condensation of the working fluid will not be considered. The second law of thermodynamics requires $\beta_2 > \beta_c > \beta_h > \beta_1$, where $\beta_i = 1/T_i$ ($i = h, c, 1$ and 2).

(4) In the two regenerating processes, the frequencies of the oscillator are constant^[9]. The frequencies of the oscillator in the two regenerating processes connecting the two isothermal processes are ω_1 and ω_2 with $\omega_1 > \omega_2$. The QST cycle plotted in the (n, ω) plane is shown in Fig.1. Assuming that the internal temperature of the working fluid in the regenerating processes changes uniformly with respect to time^[11], we have

$$\frac{dT}{dt} = \frac{d(\beta^{-1})}{dt} = \pm \frac{1}{k} \quad (2)$$

where $k = k_1 k_2$, k_1 is the time coefficient of the regeneration which is independent of temperature but dependent on the property of regenerative materials, and k_2 is the delay-time coefficient which is dependent on the phase relation between the piston of the compressor and the free displacer, inside which the regenerator is. The signs “+” and “-” correspond to the processes 4–1 and 2–3, respectively.

(5) The irreversible cold losses inside the cooler make real cold capacity far less than that obtained by an endoreversible theoretical analysis. The coefficient of cold losses ϕ_1 is defined as

$$\sum Q_i = \phi_1 Q_2 \quad (3)$$

Where Q_2 is the cold capacity obtained by the endoreversible theory^[4], and $\sum Q_i$ is the sum of the cold losses inside the cooler besides the leak Q_r .

Cycle Period

The Hamiltonian of the harmonic oscillator system S , $\hat{H}_s$, is described in the following form^[5]

$$\hat{H}_s = \omega \hat{N} = \omega \hat{a}^+ \hat{a} \quad (4)$$

where $\hat{N}$ is the number operator, $\hat{a}^+$ and $\hat{a}$ are the Bosonic creation and annihilation operators $\{\hat{N} = \hat{a}^+ \hat{a}, [\hat{a}, \hat{a}^+] = \hat{a} \hat{a}^+ - \hat{a}^+ \hat{a} = 1\}$.

In the two isothermal processes, the working fluid system S becomes an open system owing to the coupling action between S and the heat sink B_h (or the source B_c). Using the generalized master equation of an open system, and in the Heisenberg picture, one obtains the motion equation of the operator $\hat{x}$ in the system S ^[4,9,11]

$$\frac{d\hat{x}}{dt} = i(\hat{L}_s + T_{rB} \hat{L}_{sB} \hat{\rho}_B) \hat{x} - \sum_{ij} \{(\omega_{ji})^+ [\hat{x}, \hat{Q}_j] \hat{Q}_i - (\omega_{ij})^- \hat{Q}_i [\hat{x}, \hat{Q}_j]\} \quad (5)$$

where $\hat{L}_s$ is the Liouville's operator of the system S , T_{rB} represents the trace of the operator of the heat reservoir (B_h or B_c), $\hat{L}_{sB}$ represents the coupling action between the working fluid S and the heat reservoir (B_h or B_c), $\hat{\rho}_B$ is the density matrix of the reservoir (B_h or B_c), $(\omega_{ji})^+$ and $(\omega_{ij})^-$ are constant related to the matrix elements of the density matrix $\hat{\rho}_B$, $\hat{Q}_i$ and $\hat{Q}_j$ are the operators of the working fluid S and $[\cdot]$ is the quantum Poisson's bracket.

Solving equation (5) by setting $\hat{x} = \hat{N}$, $\hat{Q}_1 = \hat{a}^+$, $\hat{Q}_2 = \hat{a}$ and neglecting the disturbance of the working fluid on the reservoir ($\hat{L}_s + T_{rB} \hat{L}_{sB} \hat{\rho}_B = 0$) gives

$$\dot{n} = \frac{dn}{dt} = \frac{d \langle \hat{N} \rangle}{dt} = -a[\exp(\beta_j \omega) - 1]n - 1 \exp(f\beta_j \omega) \quad (6)$$

where a and f are both constants, $j = h$ and c correspond to the processes 1–2 and 3–4, respectively.

Substituting equation (1) into equation (6), expanding the time integral to the second order in $\beta_j \omega$ at the limit ($n \gg 1$ and $\hbar \beta_j \omega \ll 1$) gives

$$t_1 = \int_{\omega_1}^{\omega_2} \left(\frac{dn}{d\omega} / \dot{n} \right) d\omega \approx \frac{A}{\beta_h - \beta_1} \quad (7)$$

$$t_2 \approx \frac{A}{\beta_2 - \beta_c} \quad (8)$$

where t_1 and t_2 are the times of the isothermal processes 1–2 and 3–4, $A = (1/\omega_1 - 1/\omega_2)/a > 0$ is a constant which is independent of temperature but dependent on the quantum performance of the working fluid. $\beta_j \omega \ll 1$ in the paper is corresponding to $\hbar \beta_j \omega \ll 1$ in the international system.

From equation (2), the times t_3 and t_4 of the regeneration processes 2–3 and 4–1 are

$$t_3 = t_4 = k \left(\frac{1}{\beta_1} - \frac{1}{\beta_2} \right) \quad (9)$$

The cycle period time, $\tau = t_1 + t_2 + t_3 + t_4$ is

$$\tau = A \left[\left(\frac{1}{\beta_h - \beta_1} + \frac{1}{\beta_2 - \beta_c} \right) + \mu \left(\frac{1}{\beta_1} - \frac{1}{\beta_2} \right) \right] \quad (10)$$

where $\mu = 2k/A$.

Cooling Load and COP

According to the quantum mechanics theory^[13], the energy of the working fluid is

$$U = \langle \hat{H}_s \rangle = \langle \omega \hat{N} \rangle \quad (11)$$

From equation (11), the rate of change of energy is obtained by

$$\dot{U} = \dot{\omega} \langle \hat{N} \rangle + \omega \langle \dot{\hat{N}} \rangle = \dot{\omega} n + \omega \dot{n} \quad (12)$$

Comparing equation (12) with time derivative of the first law of thermodynamics, we obtain the instantaneous heat flow q and the power P in the following equations

$$q = \omega \dot{n} \quad (13)$$

$$P = -\dot{\omega} n \quad (14)$$

Substituting equation (1) into equation (13) and expanding the heat quantity integral to first order in $\beta_i \omega$ ($i = h, 1, c$ and 2) at the limit ($n \gg 1$ and $\beta_i \omega \ll 1$) gives

$$Q_1 = \int_{n_1}^{n_2} \omega dn \approx \frac{\lambda}{\beta_1} \quad (15)$$

$$Q_2 \approx \frac{\lambda}{\beta_2} \quad (16)$$

where Q_1 and Q_2 are the heat exchange quantities in the processes 1–2 and 3–4, the constant $\lambda = \ln(\omega_1 / \omega_2) > 0$ is the natural logarithm of the frequency ratio.

The real cooling quantity Q'_2 in a cycle is

$$Q'_2 = Q_2 - \sum Q_i = \frac{\phi_2 \lambda}{\beta_2} \quad (17)$$

where $\phi_2 = 1 - \phi_1$ can be obtained by the method of fitting the experimental data.

The Hamiltonians of the heat sink B_h and the heat source B_c are respectively

$$\hat{H}_h = \omega_h \hat{b}_h^\dagger \hat{b}_h \quad (18)$$

$$\hat{H}_c = \omega_c \hat{b}_c^\dagger \hat{b}_c \quad (19)$$

where ω_h, ω_c are the frequencies of the thermal phonon of the sink and the source, and $\hat{b}, \hat{b}^\dagger$ are the annihilation

and creation operators, and the subscript h, c stand for the sink and the source, respectively. Setting n_c as the thermal phonon number of the source gives

$$\dot{n}_c = -c \{ [\exp(\beta_h \omega_c) - 1] n_c - 1 \} \exp(g \beta_h \omega_c) \quad (20)$$

where c and g are constants.

$$\text{Combining equation (20) and } n_c = \frac{1}{\exp(\beta_c \omega_c) - 1}$$

gives

$$\dot{Q}_r = \omega_c \dot{n}_c = -c \omega_c \left[\frac{\exp(\beta_h \omega_c) - 1}{\exp(\beta_c \omega_c) - 1} - 1 \right] \exp(g \beta_h \omega_c) \quad (21)$$

where $\dot{Q}_r = \frac{Q_r}{\tau}$ is the rate of heat leak from the sink

to the source.

Because the source and the sink are both infinitely large and their internal relaxation is very strong, $\beta_h, \beta_c, \omega_c$ are all constants. Thus the heat flow $\dot{Q}_r$ is constant.

Combining equations (10), (15), (17) and (21), the cooling load R and the COP can be obtained in the following equations:

$$R = \frac{Q'_2 - Q_r}{\tau} = \frac{\phi_2 \lambda}{A} \left[\frac{y \beta_1}{\beta_h - \beta_1} + \frac{y \beta_1}{y \beta_1 - \beta_c} + \mu(y - 1) \right]^{-1} - \dot{Q}_r \quad (22)$$

$$\varepsilon = \frac{Q'_2 - Q_r}{(Q_1 - Q_r) - (Q'_2 - Q_r)} = \frac{\phi_2 - y \beta_1 \dot{Q}_r \tau / \lambda}{y - \phi_2} \quad (23)$$

where $y = \beta_2 / \beta_1$.

It may be proved from equations (22) and (23) that the R and the COP are of the same extreme point. Combining equations (22) and (23) with condition $(\partial R / \partial \beta_1)_y = 0$ (or $\partial \varepsilon / \partial \beta_1 = 0$) yields

$$R = \frac{\lambda \phi_2}{A} \left[\frac{y(1 + D)^2}{y - D^2} + \mu(y - 1) \right]^{-1} - \dot{Q}_r \quad (24)$$

$$\varepsilon = \frac{1}{y - \phi_2} \left\{ \phi_2 - \frac{A \dot{Q}_r}{\lambda} \left[\frac{y(1 + D)^2}{y - D^2} + \mu(y - 1) \right] \right\} \quad (25)$$

where $D = (\beta_c / \beta_h)^{0.5}$. equations (24) and (25) give the optimal relation between R and ε with parameter y . When the heat leak is ignored, i.e., $\dot{Q}_r = 0$, the optimal relation between the R and the COP can be obtained in the following equation:

$$R = \frac{\phi_2 \lambda}{A} \left[\frac{(1 + \varepsilon^{-1})(1 + D)^2}{1 + \varepsilon^{-1} - D^2} + \frac{\mu}{\varepsilon} \right]^{-1} \quad (26)$$

Using equation (24) and the condition $\partial R / \partial y = 0$ yields expression for the maximum cooling load R_m , i.e.,

$$R_m = \frac{\lambda \phi_2}{A \{ [1 + (1 + \mu^{0.5})D]^2 - \mu \}} - \dot{Q}_r \quad (27)$$

The COP corresponding to R_m is

$$\varepsilon_0 = \frac{\phi_2 \lambda - A \dot{Q}_r [(1 + D + D\mu^{0.5})^2 - \mu]}{\lambda [D(1 + D)\mu^{-0.5} + D^2 - \phi_2]} \quad (28)$$

When $\dot{Q}_r = 0$, equations (27) and (28) become

$$R_m = \frac{\lambda \phi_2}{A [1 + D + D\mu^{0.5}]^2 - \mu} \quad (29)$$

$$\varepsilon_0 = \frac{\phi_2}{D(1 + D)\mu^{-0.5} + D^2 - \phi_2} \quad (30)$$

Using equation (25) and the condition $\partial \varepsilon / \partial y = 0$, the maximum COP (ε_m) and the corresponding cooling load R_0 can be expressed as

$$\varepsilon_m = \frac{1}{y_0 - \phi_2} \left\{ \phi_2 - \frac{A \dot{Q}_r}{\lambda} \left[\mu(y_0 - 1) + \frac{y_0(1 + D)^2}{y_0 - D^2} \right] \right\} \quad (31)$$

$$R_0 = \frac{\lambda \phi_2}{A} \left[\frac{y_0(1 + D)^2}{y_0 - D^2} + \mu(y_0 - 1) \right]^{-1} - \dot{Q}_r \quad (32)$$

where

$$y_0 = [a_2 + (a_2^2 - a_1 a_3)^{0.5}] / a_1$$

$$a_1 = A \dot{Q}_r \mu \phi_2 / \lambda - A \dot{Q}_r \mu / \lambda - \phi_2 + A \dot{Q}_r (1 + D)^2 / \lambda$$

$$a_2 = D^2 (A \dot{Q}_r \mu \phi_2 / \lambda - A \dot{Q}_r \mu / \lambda - \phi_2)$$

$$a_3 = D^2 a_2 - A \dot{Q}_r \phi_2 D^2 (1 + D)^2 / \lambda$$

Discussion

The optimum cooling load versus the COP characteristics depend on λ / A , ϕ_2 , $\dot{Q}_r$, μ and β_c / β_h . The general relationship of the optimal cooling load versus the COP for a QST cooler is shown in Fig.2 and Fig.3. The curve is a loop-shaped one, thus we obtain the optimal region of a QST cooler, i.e.

$$R_0 \leq R \leq R_m, \varepsilon_0 \leq \varepsilon \leq \varepsilon_m \quad (33)$$

where R_0 , R_m and ε_0 , ε_m are determined by equation (32), (27) and (28), (31).

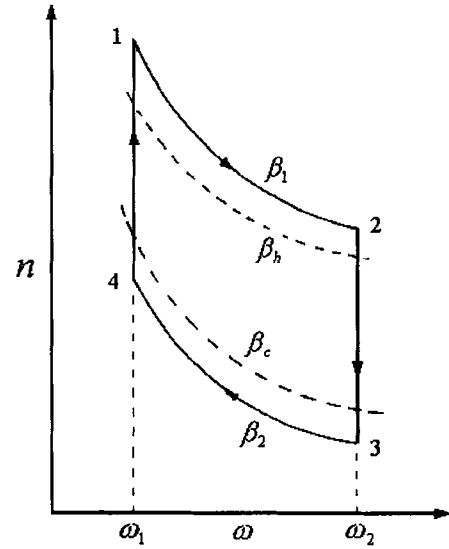


Fig.1 A QST cycle in the (ω, n) plane

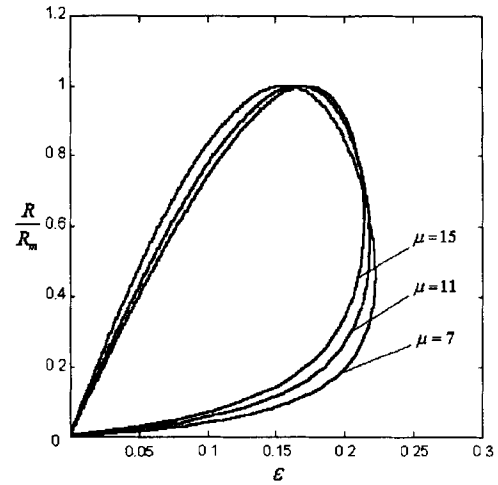


Fig.2 R/R_m versus ε characteristics: the effects of μ

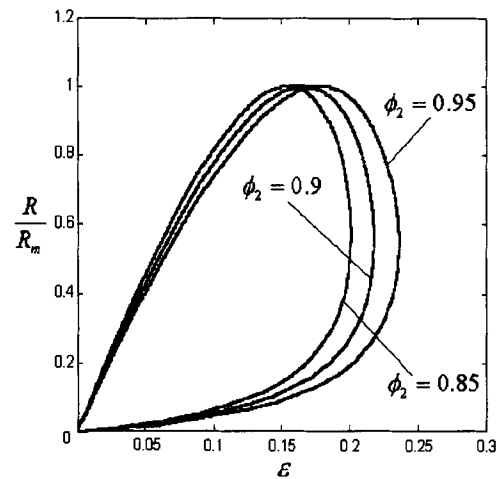


Fig.3 R/R_m versus ε characteristics: the effects of ϕ_2

Conclusion

The finite time thermodynamic model of a QST cooler has been established in this paper. It provides the application foundation of the non-equilibrium statistical mechanics to the practical ST cycles. The solution of the generalized quantum master equation of a thermal system is obtained in the Heisenberg picture. It makes the quantum picture of the coupling action between the working fluid and the source (or sink) more clear and simple.

The optimization region (or criteria) for QST cooler is obtained in the paper. The relation between the cooling load R and the COP, the maximum cooling load R_m and the corresponding COP, as well as the maximum COP and the corresponding cooling load for an irreversible QST cooler with heat leak and other irreversible losses are derived. The analysis provides a new theoretical basis for the performance evaluation and improvement of real ST cooler.

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